

# Non-relativistic Lee Model in Three Dimensional Riemannian Manifolds

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## Abstract

In this work, we construct the non-relativistic Lee model on some class of three dimensional Riemannian manifolds by following a novel approach introduced by S. G. Rajeev [1]. This approach together with the help of heat kernel allows us to perform the renormalization non-perturbatively and explicitly. For completeness, we show that the ground state energy is bounded from below for different classes of manifolds, using the upper bound estimates on the heat kernel. Finally, we apply a kind of mean field approximation to the model for compact and non-compact manifolds separately and discover that the ground state energy grows linearly with the number of bosons  $n$ .

## 1 Introduction

The Lee model, originally introduced in [2], is an exactly soluble and renormalizable model that includes the interaction between a relativistic bosonic field (“pion”) and a heavy source (“nucleon”) with one internal degree of freedom, which has two eigenvalues distinguishing “proton” and “neutron”. By heavy, we mean that the recoil of the source is neglected. Although this model is not very realistic, it reflects important features of nucleon-pion system and presents a powerful aspect that one can do the renormalization

without the use of perturbation techniques. Moreover, the complete non-relativistic version of this model that describes one heavy particle sitting at some fixed point interacting with a field of non-relativistic bosons is as important as its relativistic counterpart. It is simpler than its relativistic version because it is possible to renormalize the Hamiltonian of the system with only an additive renormalization of the mass (energy) difference of the fermions. It has been studied in a textbook by Henley and Thirring for small number of bosons from the point of view of scattering matrix [3] and there are further attempts in the literature from several point of views [4]. It is possible to look at the same problem from the point of view of the resolvent of the Hamiltonian in a Fock space formalism with arbitrary number of bosons (in fact there is a conserved quantity which allows us to restrict the problem to the direct sum of  $n$  and  $n + 1$  boson sectors). This is achieved in a very interesting unpublished paper by S. G. Rajeev [1], in which a new non-perturbative formulation of renormalization has been proposed. We are not going to review the ideas developed in there. Instead, we suggest the reader to read through the paper [1] to make the reading of this paper easier.

Following the original ideas developed in [1], we wish to extend the non-relativistic Lee model onto the Riemannian manifolds with the help of heat kernel techniques, hoping that one may understand the nature of renormalization on general curved spaces better. In this work, for the sake of simplicity we ignore the motion of the heavy particle and take its position  $a$  as a given fixed point on the manifold. The construction of the model is simply based on finding the resolvent of the regularized Hamiltonian  $H_\epsilon$  and show that a well-definite finite limit of the resolvent exists as  $\epsilon \rightarrow 0^+$  (called renormalization) with the help of heat kernel. We prove that the ground state energy for a fixed number of bosons is bounded from below, using the lower bound estimates of heat kernel for some class of Riemannian manifolds, e.g., Cartan-Hadamard manifolds (also explicitly  $\mathbb{H}^3$ ), the minimal submanifolds of  $\mathbb{R}^3$  and closed compact manifolds with nonnegative Ricci curvature. We also study the model in the mean field approximation for compact and non-compact manifolds separately and prove that the ground state energy grows linearly with the number of bosons for both classes of manifolds.

The paper is organized as follows. In the first part, we construct the model and show that the renormalization can be accomplished on Riemannian manifolds. Then, we prove that there exists a lower bound on the ground state energy. Finally, the model is examined in the mean field approximation.

## 2 The Construction of the Model

We start with the regularized Hamiltonian of the non-relativistic Lee model on a three dimensional Riemannian manifold  $(\mathcal{M}, g)$  with a cut-off  $\epsilon$ . Adopting the natural units ( $\hbar = c = 1$ ), one can write down the regularized Hamiltonian on the local coordinates  $x = (x_1, x_2, x_3) \in \mathcal{M}$

$$H_\epsilon = H_0 + H_{I,\epsilon} , \quad (1)$$

where

$$H_0 = \int_{\mathcal{M}} d_g x \, \phi^\dagger(x) \left( -\frac{1}{2m} \nabla_g^2 + m \right) \phi(x) , \quad (2)$$

$$H_{I,\epsilon} = \mu(\epsilon) \frac{1 - \sigma_3}{2} + \lambda \int_{\mathcal{M}} d_g x \, \rho_\epsilon(x, a) \left( \phi(x) \sigma_- + \phi^\dagger(x) \sigma_+ \right) . \quad (3)$$

Here,  $d_g x = \sqrt{\det g_{ij}} dx$  is the volume element and  $\nabla_g^2$  is Laplace-Beltrami operator or simply Laplacian, and  $\phi^\dagger(x)$ ,  $\phi(x)$  is the bosonic creation-annihilation operators defined on the manifold with the metric structure  $g_{ij}$ . Sometimes we shall write  $\phi_g(x)$  in order to specify which metric structure it is associated with but for now we simply write down  $\phi(x)$ . Also,  $\rho_\epsilon(x, a)$  is a family of functions which converge to the Dirac delta function  $\delta_g(x, a)$  (with the normalization  $\int_{\mathcal{M}} d_g x \, \delta_g(x, a) = 1$ ) around the point  $a$  on  $\mathcal{M}$  as we take the limit  $\epsilon \rightarrow 0^+$ . The Pauli spin matrices  $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$  and  $\sigma_3$  are regarded as a matrix representation of the fermionic creation and annihilation operators acting on  $\mathbb{C}^2$  and  $\mu(\epsilon)$  is a bare mass difference between the “proton” and “neutron” states of the two state system (“nucleon”). Its explicit form will be determined later on. Although the number of bosons is not conserved in the model, one can derive from the equations of motion that there exists a conserved quantity

$$Q = -\frac{1 - \sigma_3}{2} + \int_{\mathcal{M}} d_g x \, \phi^\dagger(x) \phi(x)$$

which takes only positive integer values. If  $Q = n \in \mathbb{Z}^+$ , we have spin-up state (“proton”) with  $n$  bosons or spin-down state (“neutron”) with  $n - 1$  bosons. We can think of the latter as a bound state of the system. If one considers the Hamiltonian without the cut-off  $\epsilon$ , it can be shown that the bound state energy diverges. Before discussing how to deal with the infinities, we must see how the infinities emerge in our model. The simplest way to realize this is just to look at the sector that contains the “neutron” or the “proton” with one boson, which corresponds to  $Q = 1$ , that is, we propose the following eigenstate ansatz [5]:

$$|u, \psi\rangle = \begin{pmatrix} \int_{\mathcal{M}} d_g x \, \psi(x) \phi^\dagger(x) |0\rangle \\ u |0\rangle \end{pmatrix} . \quad (4)$$

For simplicity we explicitly perform our calculations for compact manifolds here, but our result is also valid for non-compact manifolds, which we are interested in. If the manifold  $\mathcal{M}$  is compact, then the Laplacian has a discrete spectrum and there is a family of orthonormal complete eigenfunctions  $f_\sigma(x) \in L^2(\mathcal{M})$  [6] satisfying,

$$\begin{aligned} -\nabla_g^2 f_\sigma(x) &= \sigma f_\sigma(x) , \\ \int_{\mathcal{M}} d_g^3 x f_\sigma^*(x) f_{\sigma'}(x) &= \delta_{\sigma\sigma'} , \\ \sum_{\sigma} f_\sigma^*(x) f_\sigma(y) &= \delta_g(x, y) . \end{aligned} \quad (5)$$

In fact, one can extend these expressions to some noncompact manifolds, and they also satisfy these properties by an appropriate generalization of the measures to the continuous distributions in the sense of [7].

From the eigenvalue equation  $H|u, \psi\rangle = E|u, \psi\rangle$ , we find the set of equations in terms of the bosonic wave function  $\psi(x) = \sum_{\sigma} f_{\sigma}(x)\psi(\sigma)$

$$\psi(\sigma) = \frac{u\lambda f_{\sigma}^*(a)}{E - \left(\frac{\sigma}{2m} + m\right)} , \quad (6)$$

$$u(E - \mu) = \lambda \sum_{\sigma} f_{\sigma}(a)\psi(\sigma) . \quad (7)$$

If we substitute the equation (6) into the equation (7), we obtain

$$\mu = E - \lambda^2 \sum_{\sigma} \frac{|f_{\sigma}(a)|^2}{E - \left(\frac{\sigma}{2m} + m\right)} .$$

Expressing this equation in terms of heat kernel  $K_{s/2m}(x, x') = \langle x | e^{s\nabla_g^2/2m} | x' \rangle$ , we get

$$\mu = E + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) e^{-s(m-E)} . \quad (8)$$

Let us make a short digression on heat kernels. When we want to emphasize the metric structure  $g_{ij}$  on which the heat kernel is based, we shall use the notation  $K_{s/2m}(x, x'; g)$  throughout the paper. Although the notion of heat kernel can be defined on any Riemannian manifold, the explicit formulas only exist for some special class of manifolds, for example, Euclidean spaces  $\mathbb{R}^d$  [8] and hyperbolic spaces  $\mathbb{H}^d$  [9]. Some of the well known properties of the heat kernel on  $\mathcal{M}$  in dimensionless parameters  $t$  and  $x$  are

$$\begin{aligned} K_t(x, y) &= K_t(y, x) && \text{Symmetry property ,} \\ \frac{\partial K_t(x, x')}{\partial t} - \nabla_g^2 K_t(x, x') &= 0 && \text{Heat equation ,} \\ \lim_{t \rightarrow 0^+} K_t(x, x') &= \delta_g(x, x') && \text{Initial condition ,} \\ \int_{\mathcal{M}} d_g x K_{t_1}(x, z) K_{t_2}(z, y) &= K_{t_1+t_2}(x, y), && \text{Reproducing property ,} \end{aligned} \quad (9)$$

for  $t \geq 0$  only. If in addition  $\mathcal{M}$  is a compact manifold, we have

$$K_t(x, y) = \sum_{\sigma} e^{-t\sigma} f_{\sigma}^*(x) f_{\sigma}(y) , \quad (10)$$

which converges uniformly on  $\mathcal{M} \times \mathcal{M}$ . Again we assume that this has a proper analogous expression for non-compact manifolds we are interested in. When the manifold  $\mathcal{M}$  is a complete Riemannian manifold with Ricci curvature bounded from below then the heat kernel satisfies the stochastic completeness property [8, 10]:

$$\int_{\mathcal{M}} d_g x K_t(x, y) = 1$$

On a compact manifold stochastic completeness is always satisfied [11].

The integral in the equation (8) diverges due to the asymptotic expansion of the diagonal part of heat kernel near  $s = 0$  for any three dimensional geodesically complete manifold whose injectivity radius has a positive lower bound [12]

$$\lim_{s \rightarrow 0^+} K_{s/2m}(a, a) \sim \frac{1}{(4\pi s/2m)^{3/2}} \sum_{k=0}^{\infty} u_k(a, a) (s/2m)^k . \quad (11)$$

Here the  $u_k(a, a)$  are functions given in terms of curvature tensor of the manifold and its covariant derivatives at the point  $a$ . As a result of this, the bound state energy becomes divergent. In flat spaces, one can do the similar calculations in momentum space and find that [5]

$$\mu = E - \lambda^2 \int \frac{dp}{E - \omega(p)} \quad (12)$$

where  $\omega(p) = \frac{p^2}{2m} + m$ . This momentum integral blows up at large values of momentum in three dimensions. The problem is basically taking the integral over all momenta because our no-recoil approximation breaks down for large enough momenta. So, we introduce an ultraviolet cut-off  $\Lambda$  in the upper bound of the integral. Since large momenta means small distances, this cut-off corresponds to putting a small distance cut-off in coordinate space. Performing the calculations in coordinate space one see that small distance cut-off can be replaced with a short “time” cut-off  $\epsilon$  in the lower limit of the integral (8). Here we show that the idea of short “time” cut-off will work on Riemannian manifolds, whereas the momentum cut-off is not a natural method to use.

Therefore, we first regularize the Hamiltonian, that is, introduce the cut-off  $\epsilon$  on the lower bound of the integral and make the parameters in the Hamiltonian depend on  $\epsilon$  such that all physical quantities are independent of it. So we define

$$\mu(\epsilon) = \mu + \lambda^2 \int_{\epsilon}^{\infty} ds K_{s/2m}(a, a) e^{-s(m-\mu)} , \quad (13)$$

where  $E$  is traded with  $\mu$  which is defined as the physical energy of the composite state which consists of a boson and the attractive heavy neutron at the center. Now, using (6), (7) and (13), we get the finite expression for the bound state energy

$$E = \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(m-E)} \right]. \quad (14)$$

Here it is easy to see that  $E = \mu$  is a possible solution to this equation. Then, one can calculate the bosonic wave function  $\psi(x)$  for  $E = \mu$

$$\psi(x) = -u\lambda \int_0^\infty ds K_{s/2m}(x, a) e^{-s(m-\mu)}, \quad (15)$$

if  $x \neq a$ . Although the wave function is divergent as  $x \rightarrow a$ , it is square integrable as we will see. If we substitute (15) into our ansatz (4), we get

$$|u\psi\rangle = u \begin{pmatrix} -\frac{\lambda}{H_0 - \mu} \phi^\dagger(a) |0\rangle \\ |0\rangle \end{pmatrix}. \quad (16)$$

Normalizability of (16) can be easily seen by using the properties of heat kernel <sup>1</sup>:

$$\begin{aligned} & \lambda^2 \int_{\mathcal{M}} d_g x \int_0^\infty ds_1 ds_2 K_{s_1/2m}(x, a) K_{s_2/2m}(x, a) e^{-(s_1+s_2)(m-\mu)} \\ &= \int_0^\infty ds \left( \frac{1}{2} \int_{-s}^s dt \right) K_{s/2m}(a, a) e^{-s(m-\mu)} = \int_0^\infty s ds e^{-s(m-\mu)} K_{s/2m}(a, a), \end{aligned} \quad (17)$$

and as a result we find the normalization to be

$$\left[ 1 + \lambda^2 \int_0^\infty s ds e^{-s(m-\mu)} K_{s/2m}(a, a) \right]^{-1/2}. \quad (18)$$

This integral is finite due to the short and long time behaviour of heat kernel.

One can also consider the scattering of a boson from the “proton” at rest *on a non-compact manifold* <sup>2</sup>. The inhomogeneous Schrödinger equation  $(H - E)|u, \psi\rangle = |v, \chi\rangle$  leads to

$$\psi(\sigma) = \frac{\chi(\sigma) - \lambda u f_\sigma^*(a)}{\frac{\sigma}{2m} + m - E} \quad (19)$$

and

$$\lambda \sum_\sigma f_\sigma(a) \psi(\sigma) + u(\mu - E) = v, \quad (20)$$

If we substitute (19) into (20), we find

$$- \lambda^2 u \sum_\sigma \frac{|f_\sigma(a)|^2}{\frac{\sigma}{2m} + m - E} + u(\mu - E) = v - \lambda \sum_\sigma \frac{f_\sigma(a) \chi(\sigma)}{\frac{\sigma}{2m} + m - E}.$$

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<sup>1</sup>The same normalization can also be found by writing the operators  $\phi(a)$  in the eigenbasis  $f_\sigma(a)$  of the Laplacian.

<sup>2</sup>Compact manifolds have only discrete spectrum.

If we express the above equation in terms of heat kernel, we immediately see that the integral is divergent due to singularity near  $s = 0$ . So we must do the regularization by introducing a cut-off  $\epsilon$  in the lower limit of the integral and choose  $\mu(\epsilon)$  as above. Taking the limit  $\epsilon \rightarrow 0^+$  we get the following finite expression

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \left( \lambda^2 u \int_{\epsilon}^{\infty} ds K_{s/2m}(a, a) e^{-s(m-\mu)} - \lambda^2 u \int_{\epsilon}^{\infty} ds K_{s/2m}(a, a) e^{-s(m-E)} \right) \\ = v - \lambda \int_{\mathcal{M}} d_g x \chi(x) \left( \int_0^{\infty} ds K_{s/2m}(x, a) e^{-s(m-E)} \right). \end{aligned}$$

From the above equation, it follows that

$$\begin{aligned} u \equiv u[v, \chi] = \left[ \lambda^2 \int_0^{\infty} ds K_{s/2m}(a, a) \left( e^{-s(m-\mu)} - e^{-s(m-E)} \right) \right]^{-1} \\ \times \left[ v - \lambda \int_0^{\infty} ds \left( \int_{\mathcal{M}} d_g x K_{s/2m}(x, a) \chi(x) \right) e^{-s(m-E)} \right]. \end{aligned}$$

We can also read off  $\psi(x)$  from the equation (19) when  $x \neq a$

$$\psi(x) = \int_0^{\infty} ds e^{-s(m-E)} \int d_g y K_{s/2m}(x, y) \chi(y) - \lambda u[v, \chi] \int_0^{\infty} ds K_{s/2m}(x, a) e^{-s(m-E)}, \quad (21)$$

and  $\psi(a) = \lambda^{-1}(v - u[v, \chi](\mu - E))$  as a result of the equation (20). It is important to remind the reader that these expressions should be understood in the sense of analytic continuation in the complex  $E$ -plane to their largest domain of definition. Indeed if the real part of  $E$  is smaller than  $m$  these integrals all make sense, and the resulting expressions are just the Green's function, or the resolvents for the Laplace operator, which exists away from the positive real axis.

Up to now, we have shown that non-relativistic Lee model on a manifold is divergent and can be renormalized with the help of heat kernel by considering the problem for  $Q = 1$  sector. However, it is not clear in this formulation, how we can extend this method to the case that have arbitrary number of particles, say  $Q = n$  sectors and show that the ground state energy is bounded from below. Nevertheless, we have an alternative and powerful method which is developed by Rajeev [1]. From now on, we will follow his approach in order to construct and develop the model on a manifold for any sector.

Let us first express the regularized Hamiltonian as a  $2 \times 2$  block split according to  $\mathbb{C}^2$ :

$$H_{\epsilon} - E = \begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} d_g x \rho_{\epsilon}(x, a) \phi^{\dagger}(x) \\ \lambda \int_{\mathcal{M}} d_g x \rho_{\epsilon}(x, a) \phi(x) & H_0 - E + \mu(\epsilon) \end{pmatrix}. \quad (22)$$

Then, one can construct the regularized resolvent of this hamiltonian using an explicit

formula given by Rajeev [1]

$$R_\epsilon(E) = \frac{1}{H_\epsilon - E} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} \alpha &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \Phi_\epsilon^{-1}(E) b \frac{1}{H_0 - E} \\ \beta &= -\Phi_\epsilon^{-1}(E) b \frac{1}{H_0 - E} \\ \delta &= \Phi_\epsilon^{-1}(E) \\ b &= \lambda \int_{\mathcal{M}} d_g x \rho_\epsilon(x, a) \phi(x). \end{aligned} \quad (24)$$

Here  $E$  should be considered as a complex variable. Most importantly, the operator  $\Phi_\epsilon(E)$ , called principal operator, is given as

$$\Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \lambda^2 \int_{\mathcal{M}} d_g x d_g y \rho_\epsilon(x, a) \rho_\epsilon(y, a) \phi(x) \frac{1}{H_0 - E} \phi^\dagger(y), \quad (25)$$

Once we have a proper definition of the principal operator, all the divergences are removed since the resolvent is expressed in terms of it. We can extend the principal operator by analytic continuation to its largest domain of definition in the complex energy plane. For our purposes, we will assume that  $\Re(E) < nm + \mu$ . In fact, the energy of bound states are interesting and they satisfy the required conditions as we will see. Now, we will do the normal ordering of the operators in (25) by using the commutation relations of the operators  $\phi(x)$  and  $\phi^\dagger(x)$

$$\phi(x) \frac{1}{H_0 - E} = \int_{\mathcal{M}} d_g x \int_0^\infty ds e^{-s(H_0 - E)} K_{s/2m}(x, x') \phi(x'),$$

which can be proved simply by using an eigenfunction expansion. Then, the principal operator can be written in terms of heat kernel

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y d_g x' d_g y' \rho_\epsilon(x, a) \rho_\epsilon(y, a) \\ &\quad \times K_{s/2m}(x, x') K_{s/2m}(y, y') \phi^\dagger(y') e^{-s(H_0 + 2m - E)} \phi(x') \\ &+ \mu(\epsilon) - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g y d_g y' K_{s/2m}(y, y') \rho_\epsilon(y, a) \rho_\epsilon(y', a) e^{-s(H_0 + m - E)}. \end{aligned}$$

Since the heat kernel is a natural delta sequence, we can set  $\rho_\epsilon(x, a) = K_{\epsilon/4m}(x, a)$  without loss of generality. Hence,

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y d_g x' d_g y' K_{\epsilon/4m}(x, a) K_{\epsilon/4m}(y, a) \\ &\quad \times K_{s/2m}(x, x') K_{s/2m}(y, y') \phi^\dagger(y') e^{-s(H_0 + 2m - E)} \phi(x') \end{aligned}$$



$$+ \mu(\epsilon) - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g y d_g y' K_{s/2m}(y, y') K_{\epsilon/4m}(y, a) K_{\epsilon/4m}(y', a) e^{-s(H_0+m-E)} .$$

Using the reproducing property of the heat kernel, one can rewrite the above equation as

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{(2s+\epsilon)/4m}(x, a) K_{(2s+\epsilon)/4m}(y, a) \\ &\times \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) + \mu(\epsilon) - \lambda^2 \int_0^\infty ds K_{(s+\epsilon)/2m}(a, a) e^{-s(H_0+m-E)} . \end{aligned}$$

Shifting the variable  $s$  in the first integral by  $\epsilon/2$  and in the second integral by  $\epsilon$ , we get

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \\ &\times \phi^\dagger(x) e^{-(s-\epsilon/2)(H_0+2m-E)} \phi(y) + \mu(\epsilon) - \lambda^2 \int_\epsilon^\infty ds K_{s/2m}(a, a) e^{-(s-\epsilon)(H_0+m-E)} . \end{aligned}$$

If we take the limit  $\epsilon \rightarrow 0^+$ , the only divergence is coming from the lower limit of the second integral term due to singular behavior of the diagonal part ( $k = 0$  term in the sum) of heat kernel near  $s = 0$ , see equation (11).

One can also see that the first interaction term is actually finite due to the quite sharp bounds on the heat kernel for various classes of manifolds [8, 13]. In fact we will explicitly show later on that this term is really a finite expression by working out a bound on the spectrum of the model. Since the principal operator or resolvent is not well defined in this limit, we must now regularize the model by choosing  $\mu(\epsilon)$  exactly the same as in (13). Then, we find

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \\ &\times \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) + \mu + \lambda^2 \int_\epsilon^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(H_0+m-E)} \right] . \end{aligned}$$

Here the limit  $\epsilon \rightarrow 0^+$  is now well-defined so we have

$$\begin{aligned} \Phi(E) &= H_0 - E + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(H_0+m-E)} \right] \\ &- \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) . \quad (26) \end{aligned}$$

This is the renormalized form of the principal operator so that we have a well-defined explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator  $\Phi^{-1}(E)$ .

The spectrum of the Hamiltonian is the set of numbers  $E$  at which the resolvent does not exist (discrete spectrum) or exist but is unbounded (continuous spectrum). Thus, the continuous spectrum is that of  $H_0$  and the values of  $E$  where  $\Phi(E)$  does not have a

bounded inverse. Since, there are no poles in  $\frac{1}{H_0-E}$ , the poles corresponding to bound states must arise from those of  $\Phi^{-1}(E)$ , that is, the roots of the equation

$$\Phi(E)|\Psi\rangle = 0, \quad (27)$$

determine the poles in the resolvent, which means that the principal operator  $\Phi(E)$  determines the bound state spectrum of the theory. After we have found a root, we can determine the corresponding eigenstate of the Hamiltonian. A trivial example is the bosonic vacuum state,

$$\begin{aligned} \Phi(E)|0\rangle &= \left\{ H_0 - E + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(H_0+m-E)} \right] \right. \\ &\quad \left. - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) \right\} |0\rangle = 0, \end{aligned}$$

where the root can be easily found to be

$$E = \mu.$$

We remark here that the linear eigenvalue problem is converted into a nonlinear problem for an operator family parametrized through the energy eigenvalues after the renormalization.

The residue of the pole in the resolvent is the projection operator to the corresponding eigenspace of  $H$  [14]

$$\mathbb{P}_\mu = \frac{1}{2\pi i} \oint_{\Gamma_\mu} dE R(E), \quad (28)$$

where  $\Gamma_\mu$  is a small contour enclosing the isolated eigenvalue  $\mu$  in the complex plane. For this contour integral we need to evaluate the residue

$$\text{Res}_{E=\mu}(\Phi^{-1}(E)|0\rangle\langle 0|) = (\Phi'(\mu))^{-1}|0\rangle\langle 0|. \quad (29)$$

As a result of this calculation, we find the projection operator to be

$$\begin{aligned} \mathbb{P}_\mu &= \left[ 1 + \lambda^2 \int_0^\infty s ds e^{-s(m-\mu)} K_{s/2m}(a, a) \right]^{-1} \\ &\quad \times \begin{pmatrix} \frac{\lambda}{H_0-\mu} \phi^\dagger(a) |0\rangle\langle 0| \phi(a) \frac{\lambda}{H_0-\mu} & -\frac{\lambda}{H_0-\mu} \phi^\dagger(a) |0\rangle\langle 0| \\ -|0\rangle\langle 0| \phi(a) \frac{\lambda}{H_0-\mu} & |0\rangle\langle 0| \end{pmatrix}. \end{aligned} \quad (30)$$

Thus, one can read off the eigenvector of  $H$  corresponding to the root  $E = \mu$  from the projection operator and then find the normalizable eigenstate (16) with the correct normalization factor (18). This eigenstate (16) is the first excited state (eigenstate of neutron), not the vacuum in the whole Hilbert space  $\mathcal{B} \otimes \mathbb{C}^2$ . Also, it is easy to see that

the zero eigenvalue of the Hamiltonian corresponds to the proton state  $\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}$ . Here  $m > \mu$  for bound states since we want this model to describe the attractive interaction of such a two state system with bosons. However, it is not clear whether the proton is the state of lowest energy. There may be states which contain many bosons that have a lower energy. These questions will be answered in studying the principal operator as we will do in the next section.

One can also generalize these ideas into the relativistic regime with coupling constant renormalization, but this needs further investigations so we are not going to elaborate on this and will only consider the non-relativistic Lee model.

### 3 A Lower Bound on the Ground State Energy

After the renormalization of our model, we must look at the spectrum of the problem because there are many theories in which even after the renormalization there are still divergences that makes the spectrum not bounded from below [1]. In this section we will restrict  $E$  to the real axis. In order to give the proof that the energy  $E$  is bounded from below, following the same idea as in [1], we first split the principal operator as

$$\Phi(E) = K(E) - U(E) , \quad (31)$$

such that

$$K(E) = H_0 - E + \mu , \quad (32)$$

and

$$\begin{aligned} U(E) = & -\lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(H_0+m-E)} \right] \\ & + \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) . \end{aligned} \quad (33)$$

It follows immediately that  $K(E) \geq nm - E + \mu$ , so it is a positive definite operator from our assumption  $E < nm + \mu$ . Due to the positive definiteness of heat kernel  $K_{s/2m}(a, a) > 0$ , the difference of the two exponentials is a positive operator. As a consequence of this, the first integral term in  $U(E)$  is a positive operator and we can claim that

$$U(E) < U'(E) ,$$

where

$$U'(E) = \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) \phi^\dagger(x) e^{-s(H_0+2m-E)} \phi(y) .$$

This clearly forces

$$\Phi(E) > K(E) - U'(E) ,$$

or rewriting it as

$$\Phi(E) > K(E)^{1/2} \left(1 - \tilde{U}'(E)\right) K(E)^{1/2} , \quad (34)$$

where  $\tilde{U}'(E) = K(E)^{-1/2} U'(E) K(E)^{-1/2}$  and  $K(E)$ ,  $U'(E)$  are positive operators (so is  $\tilde{U}'(E)$ ). We will now show that by choosing  $E$  sufficiently large enough it is always possible to make the operator  $\Phi(E)$  strictly positive, hence it is invertible, and has no zeros beyond this particular value of  $E$ . Therefore, if we impose

$$\|\tilde{U}'(E)\| < 1 , \quad (35)$$

then the principal operator  $\Phi(E)$  becomes strictly positive. In order to do some estimates, we will rewrite the interaction term in terms of eigenfunctions  $f_\sigma(x)$  and shift the operator  $\phi^\dagger(x)$  to the leftmost side and the operator  $\phi(x)$  to the rightmost side

$$\begin{aligned} \tilde{U}'(E) &= \lambda^2 \sum_{\sigma_1, \sigma_2} \phi^\dagger(\sigma_1) f_{\sigma_1}(a) [H_0 + \mu + \sigma_1/2m + m - E]^{-1/2} \\ &\times [H_0 + \sigma_1/2m + \sigma_2/2m + 2m - E]^{-1} [H_0 + \mu + \sigma_2/2m + m - E]^{-1/2} \phi(\sigma_2) f_{\sigma_2}(a) , \end{aligned}$$

where  $\phi(\sigma) = \int_{\mathcal{M}} d_g x f_\sigma^*(x) \phi(x)$ . In order to convert the product of operators in the above formula into a summation of them, we will use the Feynman parametrization [15]

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1 \Pi_i^3 du_i \frac{\delta(\sum_i u_i - 1) u_1^{\alpha_1-1} u_2^{\alpha_2-1} u_3^{\alpha_3-1}}{[u_1 A_1 + u_2 A_2 + u_3 A_3]^{\alpha_1 + \alpha_2 + \alpha_3}} , \quad (36)$$

so that,

$$\begin{aligned} \tilde{U}'(E) &= \lambda^2 \sum_{\sigma_1, \sigma_2} \phi^\dagger(\sigma_1) f_{\sigma_1}(a) \frac{\Gamma(1/2 + 1/2 + 1)}{\Gamma(1/2)\Gamma(1/2)\Gamma(1)} \\ &\times \int_0^1 du_1 du_2 du_3 u_1^{\frac{1}{2}-1} u_2^{\frac{1}{2}-1} u_3^{1-1} \delta(u_1 + u_2 + u_3 - 1) \\ &\times \frac{1}{[H_0 + m + \mu - E + (u_1 + u_3)\frac{\sigma_1}{2m} + (u_2 + u_3)\frac{\sigma_2}{2m} + u_3(m - \mu)]^2} \phi(\sigma_2) f_{\sigma_2}(a) , \end{aligned}$$

or one can rewrite it as

$$\begin{aligned} \tilde{U}'(E) &= \lambda^2 \sum_{\sigma_1, \sigma_2} \phi^\dagger(\sigma_1) f_{\sigma_1}(a) \frac{\Gamma(2)}{\Gamma(1/2)^2} \int_0^1 \frac{du_1 du_2 du_3 \delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2}} \\ &\times \int_0^\infty s ds e^{-s(H_0 + m + \mu - E + u_3(m - \mu) + (u_1 + u_3)(\sigma_1/2m) + (u_2 + u_3)(\sigma_2/2m))} \phi(\sigma_2) f_{\sigma_2}(a) . \end{aligned}$$

Let us express this equation in terms of the heat kernel:

$$\tilde{U}'(E) = \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds \int_{\mathcal{M}} d_g x d_g y \int_0^1 \frac{du_1 du_2 du_3 \delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2}}$$

$$\times \phi^\dagger(x) K_{s(u_1+u_3)/2m}(x, a) K_{s(u_2+u_3)/2m}(y, a) e^{-s(H_0+\mu+m-E)} e^{-su_3(m-\mu)} \phi(y) .$$

It is easy to see that heat kernel has the following scaling property in 3 dimensions:

$$K_{\alpha^2 s}(x, y; g) = \alpha^{-3} K_s(x, y; \alpha^{-2} g) , \quad (37)$$

where  $g \rightarrow \alpha^{-2} g$  means that the metric  $g_{ij}$  is scaled by a conformal factor  $\alpha^{-2}$ . Thus we have

$$\begin{aligned} \tilde{U}'(E) &= \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds \int_{\mathcal{M}} d_{(u_1+u_3)^{-1}g} x d_{(u_2+u_3)^{-1}g} y \\ &\quad \times \int_0^1 \frac{du_1 du_2 du_3 \delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2}} \phi^\dagger(x) K_{s/2m}(x, a; (u_1 + u_3)^{-1}g) \\ &\quad \times K_{s/2m}(y, a; (u_2 + u_3)^{-1}g) e^{-s(H_0+\mu+m-E)} e^{-su_3(m-\mu)} \phi(y) . \end{aligned}$$

In addition, under the scaling of metric, the commutation relations obey the following rule

$$[\phi_{\alpha^2 g}(x), \phi_{\alpha^2 g}^\dagger(y)] = \delta_{\alpha^2 g}(x, y) , \quad (38)$$

or

$$[\alpha^{3/2} \phi_{\alpha^2 g}(x), \alpha^{3/2} \phi_{\alpha^2 g}^\dagger(y)] = \delta_g(x, y) . \quad (39)$$

This lead us to find the scaling property of the creation and annihilation operators  $\phi_{\alpha^2 g}(x) = \alpha^{-3/2} \phi_g(x)$ . Using  $\phi_{(u_1+u_3)^{-1}g}(x) = (u_1 + u_3)^{3/4} \phi_g(x)$  we define the creation and annihilation operators with respect to the new metric and obtain

$$\begin{aligned} \tilde{U}'(E) &= \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds \int_{\mathcal{M}} d_{(u_1+u_3)^{-1}g} x d_{(u_2+u_3)^{-1}g} y \\ &\quad \times \int_0^1 \frac{du_1 du_2 du_3 \delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2} (u_1 + u_3)^{3/4} (u_2 + u_3)^{3/4}} \\ &\quad \times K_{s/2m}(x, a; (u_1 + u_3)^{-1}g) K_{s/2m}(y, a; (u_2 + u_3)^{-1}g) \\ &\quad \times \phi_{(u_1+u_3)^{-1}g}^\dagger(x) e^{-s(H_0+\mu+m-E)} e^{-su_3(m-\mu)} \phi_{(u_2+u_3)^{-1}g}(y) . \end{aligned}$$

In order to give an upper bound estimate on the norm of the operator  $\tilde{U}'(E)$ , we apply the Cauchy-Schwartz inequality in the norm corresponding to the new metric and get

$$\begin{aligned} ||\tilde{U}'(E)|| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds e^{-s(nm+\mu-E)} \int_0^1 \frac{du_1 du_2 du_3 \delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2} (u_1 + u_3)^{3/4} (u_2 + u_3)^{3/4}} \\ &\quad \times \left[ \int_{\mathcal{M}} d_{(u_1+u_3)^{-1}g} x K_{s/2m}^2(x, a; (u_1 + u_3)^{-1}g) \right]^{1/2} \\ &\quad \times \left[ \int_{\mathcal{M}} d_{(u_2+u_3)^{-1}g} y K_{s/2m}^2(y, a; (u_2 + u_3)^{-1}g) \right]^{1/2} . \end{aligned}$$

Here we have replaced the term  $H_0 + m + \mu - E$  in the exponent by its minimum value  $nm + \mu - E$  and dropped the term  $e^{-su_3(m-\mu)} < 1$ . By using the reproducing property of the heat kernel, we get

$$\begin{aligned} ||\tilde{U}'(E)|| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds e^{-s(nm+\mu-E)} \int_0^1 \frac{du_1 du_2 du_3}{(u_1 u_2)^{1/2} (u_1 + u_3)^{3/4} (u_2 + u_3)^{3/4}} \\ &\times \delta(u_1 + u_2 + u_3 - 1) [K_{s/m}(a, a; (u_1 + u_3)^{-1} g)]^{1/2} [K_{s/m}(a, a; (u_2 + u_3)^{-1} g)]^{1/2}. \end{aligned}$$

Scaling back to the original variables, we finally obtain

$$\begin{aligned} ||\tilde{U}'(E)|| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds e^{-s(nm+\mu-E)} \int_0^1 \frac{du_1 du_2 du_3}{(u_1 u_2)^{1/2}} \delta(u_1 + u_2 + u_3 - 1) \\ &\times [K_{s(u_1+u_3)/m}(a, a; g)]^{1/2} [K_{s(u_2+u_3)/m}(a, a; g)]^{1/2}. \quad (40) \end{aligned}$$

It is essential here to note the behavior of the heat kernel. In most of the situations, the explicit form of the heat kernel is unknown so one should look for the estimates or upper bounds on it. Luckily, there are quite sharp upper bounds on the heat kernel for various classes of manifolds in the mathematical literature [8, 13, 16, 18]. For each class of manifolds, there are different bounds so we will consider them separately.

We will first consider Cartan-Hadamard manifolds, which are geodesically complete simply connected non-compact Riemannian manifolds with non-positive sectional curvature  $-K^2$  (for example  $\mathbb{R}^d$  and  $\mathbb{H}^d$ ). On 3 dimensional Cartan-Hadamard manifolds, we have the following upper bound on the heat kernel (see chapter 7.4 in [8] and [16])

$$K_{s/2m}(x, x) \leq \frac{C}{\min \left\{ \frac{1}{2m}, (s/2m)^{3/2} \right\}} e^{-K^2 s/2m}, \quad (41)$$

or for simplicity of our calculations, one can use the following less sharp estimate [8]

$$K_{s/2m}(x, x) \leq \frac{C}{(s/2m)^{3/2}}, \quad (42)$$

for all  $x \in \mathcal{M}$  and  $s > 0$ , and  $C$  is a dimensionless constant. This estimate (42) can be extended also to the minimal submanifolds of  $\mathbb{R}^3$  [8]. Then the strict positivity of the principal operator after taking the integrals leads to the following inequality

$$||\tilde{U}'(E)|| < n \lambda^2 m^{3/2} C \frac{\pi^{3/2} \Gamma(2) \Gamma(1/4)^2}{\Gamma(1/2)^2 \Gamma(3/4)^2} (nm + \mu - E)^{-1/2} < 1.$$

This implies the inequality for  $E$  or the opposite inequality for the ground state energy written in ordinary units

$$E < nmc^2 + \mu c^2 - n^2 \frac{\tilde{C}^2 \lambda^4 m^3}{\hbar^6}, \quad E_{gr} \geq nmc^2 + \mu c^2 - n^2 \frac{\tilde{C}^2 \lambda^4 m^3}{\hbar^6}, \quad (43)$$

where

$$\tilde{C} = C \frac{\pi^{3/2} \Gamma(2) \Gamma(1/4)^2}{\Gamma(1/2)^2 \Gamma(3/4)^2}.$$

Secondly, as an explicit application, we consider the 3 dimensional hyperbolic space  $\mathbb{H}^3$ .

The heat kernel in  $\mathbb{H}^3$  is explicitly known [9] and it is in natural units

$$K_{s/2m}(x, y) = \frac{1}{R^3} \frac{d(x, y)/R}{\sinh(d(x, y)/R)} \frac{e^{-\frac{s}{2mR^2} - \frac{md^2(x, y)}{2s}}}{(4\pi \frac{s}{2mR^2})^{3/2}}, \quad (44)$$

where  $R$  is the length scale and  $d(x, y)$  is the geodesic distance between two points on  $\mathbb{H}^3$ .

Again we impose the strict positivity condition on the principal operator and get

$$\begin{aligned} \|\tilde{U}'(E)\| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^1 \frac{du_1 du_2}{(u_1 u_2)^{1/2} (1-u_1)^{3/4} (1-u_2)^{3/4}} \\ &\times \int_0^\infty \frac{s ds e^{-s(nm+\mu-E)} e^{-s(1-u_1)/2mR^2} e^{-s(1-u_2)/2mR^2}}{(4\pi s/m)^{3/2}} < 1. \end{aligned}$$

Using the fact that  $e^{-s(1-u_1)/2mR^2} < 1$  and then taking the integrals, one can immediately see that this implies

$$E_{gr} \geq nmc^2 + \mu c^2 - n^2 \frac{\lambda^4 D^2 m^3}{\hbar^6}, \quad (45)$$

where the constant  $D$  is defined as

$$D = \frac{\Gamma(2) \Gamma(1/4)^2}{\Gamma(1/2)^2 \Gamma(3/4)^2 2^3}.$$

Finally, we apply our method to the closed compact manifolds with Ricci curvature bounded from below by  $-\kappa$ . The estimate for this class of manifolds on the heat kernel [13, 18] is given by

$$K_{s/2m}(a, a) \leq \frac{1}{V(\mathcal{M})} + A(s/2m)^{-3/2}, \quad (46)$$

where  $A = A(V(\mathcal{M}), \kappa, d)$  is an explicitly calculable constant which depends on the volume  $V(\mathcal{M})$  of the manifold, the lower bound  $\kappa$  on the Ricci curvature and the diameter  $d$  of the manifold. Using this estimate, it follows that

$$\begin{aligned} \|\tilde{U}'(E)\| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s ds e^{-s(nm+\mu-E)} \int_0^1 \frac{du_1 du_2}{(u_1 u_2)^{1/2}} \\ &\times \left[ \frac{1}{V(\mathcal{M})^{1/2}} + A^{1/2}(s(1-u_1)/m)^{-3/4} \right] \left[ \frac{1}{V(\mathcal{M})^{1/2}} + A^{1/2}(s(1-u_2)/m)^{-3/4} \right]. \end{aligned}$$

Integrating with respect to  $u_1, u_2$  and  $s$ , we have

$$\|\tilde{U}'(E)\| < n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \left[ \frac{4}{V(\mathcal{M})} \frac{1}{(nm + \mu - E)^2} + \frac{4A^{1/2} m^{3/4} \pi^{1/2} \Gamma(1/4)}{V^{1/2} \Gamma(3/4)} \right]$$

$$\times \frac{1}{(nm + \mu - E)^{5/4}} + \frac{Am^{3/2}\pi\Gamma(1/4)^2}{\Gamma(3/4)^2} \frac{1}{(nm + \mu - E)^{1/2}} \Big].$$

In order to get explicit estimates, let us put some further assumption  $nm + \mu - E > \mu$ .

Then, we find

$$\begin{aligned} \|\tilde{U}'(E)\| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \left[ \frac{4}{V(\mathcal{M})\mu^{3/2}} + \frac{4A^{1/2}m^{3/4}\pi^{1/2}\Gamma(1/4)}{\mu^{3/4}V(\mathcal{M})^{1/2}\Gamma(3/4)} + \frac{Am^{3/2}\pi\Gamma(1/4)^2}{\Gamma(3/4)^2} \right] \\ &\times \frac{1}{(nm + \mu - E)^{1/2}}. \end{aligned}$$

Now if we impose the strict positivity of the principal operator we find similarly

$$E_{gr} \geq nmc^2 + \mu c^2 - n^2 \lambda^4 F^2, \quad (47)$$

where

$$F = \frac{\Gamma(2)}{\Gamma(1/2)^2} \left[ \frac{4}{V(\mathcal{M})\mu^{3/2}} + \frac{4A^{1/2}m^{3/4}\pi^{1/2}\Gamma(1/4)}{\mu^{3/4}V(\mathcal{M})^{1/2}\Gamma(3/4)} + \frac{Am^{3/2}\pi\Gamma(1/4)^2}{\Gamma(3/4)^2} \right].$$

Therefore, lower bound on the ground state energies for different classof manifolds (43), (45) and (47) are of almost the same form up to a constant factor so the form of the lower bound has a general character. It is also worth pointing out that the form of the lower bound on the ground state energy is same as in the case for the flat space  $\mathbb{R}^3$  [1, 17]. From the general form of the lower bounds, we conclude that for each sector with a fixed number of bosons, there exists a ground state. However, the ground state energy bound that we have found diverges quadratically as the number of bosons increases. In other words, these estimates with our present analysis is not good enough to prove the existence of the thermodynamic limit. To attack this problem we will study the thermodynamic limit of the model by mean field approximation.

## 4 Mean Field Approximation of the Model

In the limit of large number of bosons  $n \rightarrow \infty$ , one expects that all the bosons have the same wave function  $u(x)$  and mean field approximation is valid, as in the case of flat spaces. Therefore, one can introduce the following mean field ansatz for  $n$  particle state on a Riemannian manifold

$$|u\rangle = \frac{1}{\sqrt{n!}} \int_{\mathcal{M}} d_g x_1 d_g x_2 \cdots d_g x_n u(x_1) u(x_2) \cdots u(x_n) \phi^\dagger(x_1) \phi^\dagger(x_2) \cdots \phi^\dagger(x_n) |0\rangle, \quad (48)$$

with the normalization

$$\|u(x)\|^2 = \int_{\mathcal{M}} |u(x)|^2 d_g x = 1, \quad (49)$$



where  $|0\rangle$  denotes the vacuum state. In the mean field approximation the operators are usually approximately replaced by their expectation values in this state ( $\langle f(u) \rangle \approx f(\langle u \rangle)$ ) so the expectation value of the principal operator by applying the mean field ansatz becomes

$$\begin{aligned} \phi(E, u) = & nh_0(u) - E + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) [e^{-s(m-\mu)} - e^{-s(nh_0(u)+m-E)}] \\ & - n\lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x d_g y K_{s/2m}(x, a) K_{s/2m}(y, a) u^*(x) e^{-s(nh_0(u)+2m-E)} u(y) , \end{aligned} \quad (50)$$

called principal function and we have defined

$$h_0(u) = \int_{\mathcal{M}} d_g x \left( \frac{|\nabla_g u(x)|^2}{2m} + m|u(x)|^2 \right) . \quad (51)$$

However, the exact value of the expectation value of the principal operator is given in terms of cummulant expansion theorem [19]. Therefore, in order to write the above formula (50), we have to further assume that the corrections coming from the higher order cummulants are negligibly small and indeed we will see that this assumption is justified for the particular solution we will find.

Now, we must solve the equation  $\phi(E, u) = 0$  (due to (27)) in order to find the spectrum of the problem and solve  $E$  as a function of  $u(x)$ , which gives the smallest possible value of  $E$  with the constraint (49). Hence, one can try to write  $E$  as a functional of  $u(x)$  from the equation  $\phi(E, u) = 0$  and apply the variational methods to minimize  $E$ .

However, there is no simple way to solve this variational problem . So, we follow a different method essentially the one suggested in [1]. For this purpose, let us introduce a new variable  $\chi$  and new wave function  $v(x)$

$$\begin{aligned} \chi &= nh_0(u) - E \\ v(x) &= [2m(2m + \chi)]^{-3/4} u(x) , \end{aligned} \quad (52)$$

such that the new wave functions  $v(x)$  are normalized with respect to the new metric  $\tilde{g}_{ij} = [2m(2m + \chi)]g_{ij}$

$$\int_{\mathcal{M}} d_{\tilde{g}} x |v(x)|^2 = \int_{\mathcal{M}} d_g x |u(x)|^2 = 1 . \quad (53)$$

We also define a new dimensionless parameter  $s' = (2m + \chi)s$ , and using the scaling property of heat kernel we find

$$\chi + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) [e^{-s(m-\mu)} - e^{-s(\chi+m)}]$$

$$= n\lambda^2(2m)^{3/2}(2m + \chi)^{1/2} \int_0^\infty ds' \left| \int_{\mathcal{M}} d\tilde{g}x K_{s'}(x, a; \tilde{g}) v(x) \right|^2 e^{-s'} . \quad (54)$$

One can prove now that the left hand side as a function of the variable  $\chi$  is an increasing function and it is obvious that the righthand side is positive. Therefore, the left hand side is minimum when  $\chi = -\mu$ , which is attained when the right hand side becomes zero (so  $\chi \geq -\mu$ ). Let us denote the inverse function of the left hand side as  $f_1(nU)$ , that is,

$$\chi = f_1(nU) , \quad (55)$$

and express  $\chi$  in terms of the energy  $E$  and the function  $v(x)$ ,

$$\chi = n[\chi + 2m]K[v] + nm - E, \quad \text{or} \quad (56)$$

$$E = nm + 2mnK[v] + (nK[v] - 1)f_1(nU) , \quad (57)$$

where  $K[v] = \int d\tilde{g}x |\nabla_{\tilde{g}} v(x)|^2$ , which is considered to be the parameter of the model because it is the variable we can control and we may use many trial functions  $v(x)$  and they can be scaled to any desired value. If we assume that  $nK[v] > 1$ , then the energy  $E$  (57) is minimized when  $f_1(nU)$  is minimized which happens for  $U[v] = 0$ . Since  $\chi = f_1(nU) \geq -\mu$ , we have

$$E \geq nm + \mu \quad \text{if} \quad nK[v] > 1 . \quad (58)$$

On the other hand, if  $K[v]$  is small enough, i.e.,  $nK[v] < 1$ , we also see that the minimum of the energy is attained with the reversed sign of the last term. In that case, we should find an upper bound for  $f_1(nU)$  which is expressed in terms of the kinetic energy functional  $K[v]$ . In order to discuss the case  $nK[v] < 1$  properly, we will separate our calculations for compact and non-compact manifolds.

Let us first consider the case for compact manifolds. In order to achieve our aim, we will go back to the original variable  $u(x)$  and the parameter  $s$  in the equation (54)

$$\begin{aligned} \chi + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(\chi+m)} \right] \\ = n\lambda^2 \int_0^\infty ds \left| \int_{\mathcal{M}} d_g x K_{s/2m}(x, a; g) u(x) \right|^2 e^{-s(\chi+2m)} = nU[u] . \end{aligned} \quad (59)$$

If we assume that the mean field approximation gives us a reliable equality, we can find the solution for  $\chi$  given all the other parameters. Note that the right hand side of (59) is a decreasing function of  $\chi$  whereas the left hand side is an increasing one. Hence there is always a unique solution which defines the inverse function for a given  $u(x)$ . It is quite

easy to see by a graphical construction that if we replace the left hand side by a smaller function of  $\chi$ , and the right hand side by a larger function of  $\chi$ , then we get an upper bound for the inverse function. We note first that

$$\int_{\mathcal{M}} d_g x K_{s/2m}(x, a; g) u(x) = \sum_{\sigma} e^{-\frac{s\sigma}{2m}} f_{\sigma}(a) u(\sigma) ,$$

where  $u(\sigma) = \int_{\mathcal{M}} d_g x f_{\sigma}^*(x) u(x)$ . If we define the following functions

$$f'_{\sigma}(a) = \begin{cases} \frac{2m f_{\sigma}(a)}{\sqrt{\sigma}} & \text{if } \sigma \neq 0 \\ f_0(a) & \text{if } \sigma = 0, \end{cases} \quad (60)$$

and

$$u'(\sigma) = \begin{cases} \frac{\sqrt{\sigma} u(\sigma)}{2m} & \text{if } \sigma \neq 0 \\ u(0) & \text{if } \sigma = 0, \end{cases} \quad (61)$$

we can write

$$\begin{aligned} \left| \int_{\mathcal{M}} d_g x K_{s/2m}(x, a; g) u(x) \right|^2 &= \left| \sum_{\sigma} e^{-\frac{s\sigma}{2m}} f'_{\sigma}(a) u'(\sigma) \right|^2 \\ &\leq \sum_{\sigma} e^{-\frac{s\sigma}{m}} |f'_{\sigma}(a)|^2 \sum_{\sigma} |u'(\sigma)|^2 \leq \left( |f_0(a)|^2 + 2m \sum'_{\sigma} \frac{e^{-\frac{s\sigma}{m}}}{\sigma/2m} |f_{\sigma}(a)|^2 \right) \left( |u(0)|^2 + \frac{K[u]}{2m} \right) , \end{aligned}$$

where  $\sum'_{\sigma}$  is the sum which excludes the zero mode ( $\sigma = 0$ ),  $K[u] = \frac{1}{2m} \int_{\mathcal{M}} d_g x |\nabla_g u(x)|^2$ , and we have used Cauchy-Schwartz inequality. Using  $|f_0(a)|^2 = 1/V(\mathcal{M})$  and  $|u(0)|^2 \leq 1$ , we find

$$U[u] \leq \left( 1 + \frac{K[u]}{2m} \right) \Omega , \quad (62)$$

where

$$\begin{aligned} \Omega &= \frac{\lambda^2}{\chi + 2m} \left[ \frac{1}{V(\mathcal{M})} + 2m \int_0^{\infty} ds \left( 1 - e^{-s(\chi+2m)/2} \right) \right. \\ &\quad \left. \times \left( K_{s/2m}(a, a; g) - \lim_{s \rightarrow \infty} K_{s/2m}(a, a; g) \right) \right] . \quad (63) \end{aligned}$$

Here the sum which excludes the zero mode corresponds to subtraction of the large  $s$  behavior of heat kernel. Using  $K[u] = (2m + \chi)K[v]$ , the inequality becomes

$$\begin{aligned} \chi + \mu &\leq \frac{n\lambda^2}{\chi + 2m} \left[ \frac{1}{V(\mathcal{M})} + 2m \int_0^{\infty} ds \left( 1 - e^{-s(\chi+2m)/2} \right) \right. \\ &\quad \left. \times \left( K_{s/2m}(a, a; g) - \lim_{s \rightarrow \infty} K_{s/2m}(a, a; g) \right) \right] \left[ 1 + \frac{(\chi + 2m)}{2m} K[v] \right] . \end{aligned}$$

Using the upper bound estimate of the heat kernel for closed compact manifolds with Ricci curvature bounded from below by  $-\kappa$  and taking the integral with respect to  $s$ , we find that

$$\chi + \mu \leq \frac{n\lambda^2}{\chi + 2m} \left[ 1 + \left( \frac{\chi + 2m}{2m} \right) K[v] \right] \left[ \frac{1}{V(\mathcal{M})} + \sqrt{2\pi}(2m)^{5/2} A(\chi + 2m)^{1/2} \right] \quad (64)$$

If we now introduce the variables  $z = \chi + 2m$  and  $A' = \sqrt{2\pi}(2m)^{5/2} A$  for simplicity of notation, we find the following inequality

$$z - (2m - \mu) \leq \frac{\lambda^2}{z} \left( n + \frac{z}{2m} \right) \left( A' z^{1/2} + \frac{1}{V(\mathcal{M})} \right). \quad (65)$$

We look for a systematic expansion of  $z$  in  $n$  by allowing a fractional power. In this case we see that if we substitute the following asymptotic expansion as  $n \rightarrow \infty$ ,  $z \sim B_1 n^{\nu_1} + B_2 n^{\nu_2} + B_3 n^{\nu_3} + \dots$ , where the consecutive powers  $\nu_1, \nu_2, \dots$  decrease, we find that this asymptotic expansion is an upper bound for this inequality as long as the coefficients are to be chosen as  $B_1 = (A'\lambda^2)^{2/3}$ ,  $B_2 = \frac{1}{V(\mathcal{M})} \left( \frac{\lambda}{A'} \right)^{2/3} + \frac{1}{2m} (A'\lambda^2)^{4/3}$  and  $B_3 = 2m - \mu + \frac{\lambda^2}{2mV(\mathcal{M})}$ , where  $\nu_1 = 2/3$ ,  $\nu_2 = 1/3$ , and  $\nu_3 = 0$ . Hence we get an upper bound on the inverse function  $f_1(nU)$  as an asymptotic series in powers of  $n$ , in the spirit of mean field approximation. This upper bound can be put back into the energy equation and then we find the following lower bound on the ground state energy,

$$E_{gr} \geq nm + 2mnK[v] - (1 - nK[v]) \left( B_1 n^{2/3} + B_2 n^{1/3} + B_3 - 2m + \dots \right) \quad (66)$$

We note now that the behavior of  $K[u]$  for large  $n$  is found from the scaling law,  $K[u] = (2m + \chi)K[v] \sim n^{-1/3}$ . This in turn justifies our use of the mean field approximation since higher order derivatives are then negligible, and the solution is approaching to an essentially constant function on the manifold as  $n$  gets larger. This proves that for a compact manifold the energy is actually bounded from below by a much milder behavior and there is a nice thermodynamic limit since the energy per particle  $E/n$  will approach to the mass (rest mass energy) as  $n \rightarrow \infty$ . The same conclusion can also be drawn for non-compact manifolds as we will see.

Now we will consider the mean field approximation of the model for non-compact manifolds. We again assume that the eigenfunction expansion in compact manifolds can be generalized to the non-compact manifolds. Then, one can use the above method for the non-compact manifolds as well. However, we shall try to find the ground state energy in the mean field approximation in another way. Therefore, we first go back to the

equation:

$$\begin{aligned} \frac{1}{(\chi + 2m)^{1/2}} \left\{ \chi + \mu + \lambda^2 \int_0^\infty ds K_{s/2m}(a, a) \left[ e^{-s(m-\mu)} - e^{-s(\chi+m)} \right] \right\} \\ = n\lambda^2(2m)^{3/2} \int_0^\infty ds' \left| \int_{\mathcal{M}} d\tilde{g}x K_{s'}(x, a; \tilde{g}) v(x) \right|^2 e^{-s'} \equiv n U[v] , \end{aligned}$$

and using the generalized eigenfunction expansions, we find

$$U[v] \leq K[v] \Omega , \quad (67)$$

where

$$\Omega = \lambda^2(2m)^{3/2} \int_0^\infty ds' \left( 1 - e^{-s'/2} \right) K_{s'}(a, a; \tilde{g}) , \quad (68)$$

The explicit form of the inverse function  $f_1(nU)$  is too difficult to find so one can estimate it. In order to do this, let us first notice that

$$f_2^{-1}(\chi) < f_1^{-1}(\chi) \quad \text{then} \quad f_1(nU) < f_2(nU).$$

For this purpose, we use the simplest possible function as  $f_2^{-1}(\chi)$

$$f_2^{-1}(\chi) = \frac{\chi + \mu}{(\chi + 2m)^{1/2}} .$$

Then we can replace  $f_1(nU)$  with something bigger, and its argument with something even bigger. Moreover,  $f_2(u)$  is dominated by a simpler function

$$f_2(u) < u^2 + 2m - 2\mu ,$$

which is a very crude bound, but easy to work with. Using the upper bound for  $U[v]$ , we get

$$\chi = f_1(nU) < f_2(u) < n^2 U[v]^2 + 2m - 2\mu < n^2 (K[v])^2 \Omega^2 + 2m - 2\mu . \quad (69)$$

Hence,

$$E \geq nm + 2mnK[v] - (1 - nK[v]) (n^2 K[v]^2 \Omega^2 + 2m - 2\mu) , \quad (70)$$

where the polynomial  $(4m - 2\mu)y - \Omega^2 y^2 + \Omega^2 y^3$  *never becomes negative* within the range  $0 \leq y = nK[v] \leq 1$  if

$$\Omega^2 < 16m - 8\mu . \quad (71)$$

In this case, the minimum is achieved when  $y = 0$ . This means that the functional  $nK[v] \rightarrow 0$  by a proper family of functions. This is why it is consistent to ignore the higher order cummulants if we choose an arbitrarily slowly varying family for the function

$v(x)$ . Therefore one can conclude that the ground state energy for non-compact manifolds when  $nK[v] < 1$

$$E_{gr} \geq nm - 2(m - \mu) . \quad (72)$$

The condition on  $\Omega$  may imply some restrictions on the coupling constant  $\lambda$ . If we go back to the definition of  $\Omega$  and scaling back again to the usual geometric variables, we find

$$\Omega = \lambda^2 (\chi + 2m)^{-1/2} \int_0^\infty ds \left( 1 - e^{-s(\chi+2m)/2} \right) K_{s/2m}(a, a; g) . \quad (73)$$

Since there is a nice sharp upper bound on the heat kernel for Cartan-Hadamard manifolds and minimal submanifolds of  $\mathbb{R}^3$  (42), we can find an upper bound on  $\Omega$  and the restriction (71) gives the following upper bound on the coupling constant.

$$\lambda < \frac{(16m - 8\mu)^{1/4}}{(2\pi)^{1/4} C^{1/2} (2m)^{3/4}} . \quad (74)$$

Now, let us calculate explicitly the function  $\Omega$  for the hyperbolic manifold  $\mathbb{H}^3$ , which belongs to the class of Cartan-Hadamard manifolds. Using the result (44), we obtain

$$\Omega = \lambda^2 (\chi + 2m)^{-1/2} (4\pi/2m)^{-3/2} 2\sqrt{\pi} \left\{ \sqrt{\frac{\chi + 2m}{2} + \frac{1}{2mR^2}} - \sqrt{\frac{1}{2mR^2}} \right\} . \quad (75)$$

Then, one can easily find the upper bound on the coupling constant from the restriction on  $\Omega$

$$\lambda < 2\sqrt{4\pi(2m - \mu)}(2m)^{-3/4} \left\{ \sqrt{\frac{2m - \mu}{2} + \frac{1}{2mR^2}} - \sqrt{\frac{1}{2mR^2}} \right\}^{-1/2} . \quad (76)$$

Therefore, similar to the case for compact manifolds, we have shown that the leading behavior of the system varies linearly with the number of bosons  $n$ , which leads to a nice thermodynamic limit on non-compact manifolds.

## 5 Conclusion

In this paper, we considered the non-relativistic Lee model on various class of Riemannian manifolds inspired from the work in [1]. This method allows us to renormalize the model non-perturbatively. It has been also shown that the heat kernel plays a key role in the renormalization procedure and help us to find a lower bound on the ground state energy due to the sharp bound estimates on it for several class of manifolds. Finally, we studied the mean field approximation and showed that there exist a nice thermodynamic limit of the model for compact and non-compact manifolds.

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